Example 11-10: Partial Fraction Expansion

As an example of a case where long division is required, consider the Fourier transform

$$Y(j\omega) = \frac{1-\omega^2+j\omega}{2-\omega^2+3j\omega} \tag{11.72}$$

The first step is to express $Y(j\omega)$ as a ratio of polynomials in $j\omega$. We can do this by noting that $(j\omega)^k = \omega^k j^k$ so if we replace ω^k by $(j\omega)^k/j^k$ we will not change the value of the expression, but we will obtain a rational function in terms of powers of $(j\omega)$. In the example above, we get

$$\begin{aligned} Y(j\omega) &= \frac{1-(j\omega)^2/j^2+j\omega}{2-(j\omega)^2/j^2+3j\omega} \\ &= \frac{1+(j\omega)+(j\omega)^2}{2+3(j\omega)+(j\omega)^2} \end{aligned}$$

Now we are ready to do the polynomial long division as shown below

$$\begin{array}{r|l} & 1 \\ \hline (j\omega)^2+3(j\omega)+2 & (j\omega)^2+\ (j\omega)+1 \\ & (j\omega)^2+3(j\omega)+2 \\ \hline & -2(j\omega)-1 \end{array}$$

With the quotient and remainder from the long division, we can now express $Y(j\omega)$ as

$$Y(j\omega) = 1 - \frac{1+2(j\omega)}{2+3(j\omega)+(j\omega)^2} \tag{11.73a}$$

$$= 1 - \frac{1+2j\omega}{(1+j\omega)(2+j\omega)} \tag{11.73b}$$

The second term of $Y(j\omega)$ in (11.73b) can be decomposed into a sum via the partial fraction expansion method

$$-\frac{1+2j\omega}{(1+j\omega)(2+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega} \tag{11.74}$$

In order to find A, we multiply both sides by $(1+j\omega)$ to obtain

$$-\frac{(1+2j\omega)(1+j\omega)}{(1+j\omega)(2+j\omega)} = \frac{A(1+j\omega)}{1+j\omega} + \frac{B(1+j\omega)}{2+j\omega}$$

Next we cancel the factors $(1+j\omega)$ wherever possible

$$-\frac{1+2j\omega}{(2+j\omega)} = A + \frac{B(1+j\omega)}{2+j\omega}$$

Now we can substitute $(j\omega) = -1$ on both sides to eliminate the B term

$$-\frac{1-2}{2-1} = A + \frac{0}{2-1}$$

and we get $A = 1$. Similarly, we can find $B = -3$ by multiplying both sides of (11.74) by $(2+j\omega)$ and evaluating at $j\omega = -2$. The final additive representation for $Y(j\omega)$ in (11.72) is

$$Y(j\omega) = 1 + \frac{1}{1+j\omega} - \frac{3}{2+j\omega}$$

Recalling that the inverse Fourier transform of 1 is $\delta(t)$, it follows that the corresponding time function is

$$y(t) = \delta(t) + e^{-t}u(t) - 3e^{-2t}u(t)$$

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