Example 3-2: Spectrum of a Product

For the special case of a beat signal formed as the product of two sinusoids at 5 Hz and $\frac{1}{2}$ Hz

$$x(t) = \cos(\pi t)\sin(10\pi t) \tag{3.8}$$

it is necessary to rewrite $x(t)$ as a sum before its spectrum can be defined. The following technique for doing this relies on the inverse Euler formula

$$
\begin{aligned}
x(t) &= \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)\left(\frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\right) \\
&= \tfrac{1}{4}e^{-j\pi/2}e^{j11\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j9\pi t} \\
&\quad - \tfrac{1}{4}e^{-j\pi/2}e^{-j9\pi t} - \tfrac{1}{4}e^{-j\pi/2}e^{-j11\pi t} \\
&= \tfrac{1}{2}\cos(11\pi t - \pi/2) + \tfrac{1}{2}\cos(9\pi t - \pi/2)
\end{aligned}
\tag{3.9}
$$

From the second and third lines of this derivation, it is obvious that there are four terms in the additive combination, and the four spectrum components are at frequencies $\pm 11\pi$ and $\pm 9\pi$ rad/sec which convert to 5.5, 4.5, -4.5, and -5.5 Hz. It is worth noting that neither of the original frequencies (5 Hz and $\frac{1}{2}$ Hz) used to define $x(t)$ in (3.8) are in the spectrum. ■