Example 3-7: Fourier Series without Integration

Determine the Fourier Series coefficients of the signal:

$$x(t) = \sin^3(3\pi t)$$

Solution: There are two ways to get the a_k coefficients: plug $x(t)$ into the Fourier integral (3.26), or use the inverse Euler formula to expand $x(t)$ into a sum of complex exponentials. It is far easier to use the latter approach. Using the inverse Euler formula for $\sin(\cdot)$, we get the following expansion of the sine-cubed function:

$$\begin{aligned} x(t) &= \left(\frac{e^{j3\pi t} - e^{-j3\pi t}}{2j}\right)^3 \\ &= \frac{1}{-8j}\left(e^{j9\pi t} - 3e^{j6\pi t}e^{-j3\pi t} + 3e^{j3\pi t}e^{-j6\pi t} - e^{-j9\pi t}\right) \\ &= \frac{j}{8}e^{j9\pi t} + \frac{-3j}{8}e^{j3\pi t} + \frac{3j}{8}e^{-j3\pi t} + \frac{-j}{8}e^{-j9\pi t} \end{aligned} \tag{3.28}$$

We see that (3.28) contains four frequencies: $\omega = \pm 3\pi$ and $\omega = \pm 9\pi$ rad/s. Since $\gcd(3\pi, 9\pi) = 3\pi$, the fundamental frequency is $\omega_0 = 3\pi$ rad/sec. The Fourier Series coefficients are indexed in terms of the fundamental frequency, so

$$a_k = \begin{cases} 0 & \text{for } k = 0 \\ \mp j\frac{3}{8} & \text{for } k = \pm 1 \\ 0 & \text{for } k = \pm 2 \\ \pm j\frac{1}{8} & \text{for } k = \pm 3 \\ 0 & \text{for } k = \pm 4, \pm 5, \pm 6, \ldots \end{cases} \tag{3.29}$$

This example shows that it is not always necessary to evaluate an integral to obtain the $\{a_k\}$ coefficients. ■

McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.