Example 7-5: Convolution via $H(z)X(z)$

The z-transform method can be used to convolve the following signals:

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$
$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

The z-transforms of the sequences $x[n]$ and $h[n]$ are:

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$
$$\text{and} \quad H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

Both $X(z)$ and $H(z)$ are polynomials in z^{-1}, so we can compute the z-transform of the convolution by multiplying these two polynomials, i.e.,

$$\begin{aligned} Y(z) = H(z)X(z) = \\ &(1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4}) \\ &= z^{-1} + (-1+2)z^{-2} + (1-2+3)z^{-3} \\ &\quad + (-1+2-3+4)z^{-4} \\ &\quad + (-2+3-4)z^{-5} + (-3+4)z^{-6} + (-4)z^{-7} \\ &= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7} \end{aligned}$$

Since the coefficients of any z-polynomial are just the sequence values, with their position in the sequence being indicated by the power of (z^{-1}), we can "inverse transform" $Y(z)$ to obtain

$$\begin{aligned} y[n] = \delta[n-1] + \delta[n-2] + 2\delta[n-3] + 2\delta[n-4] \\ - \; 3\delta[n-5] + \delta[n-6] - 4\delta[n-7] \end{aligned}$$

Now we look at the convolution sum for computing the output. If we write out a few terms, we can detect a pattern that is similar to the z-transform polynomial multiplication.

$$\begin{aligned} y[0] &= h[0]x[0] = 1(0) = 0 \\ y[1] &= h[0]x[1] + h[1]x[0] = 1(1) + 2(0) = 1 \\ y[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] \\ &= 1(-1) + 2(1) + 3(0) = 1 \\ y[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0] \\ &= 1(1) + 2(-1) + 3(1) = 2 \\ y[4] &= h[0]x[4] + h[1]x[3] + h[2]x[2] + h[3]x[1] \\ &= 1(-1) + 2(1) + 3(-1) + 4(1) = 2 \\ \vdots &= \quad \vdots \end{aligned}$$

Notice how the index of $h[k]$ and the index of $x[n-k]$ sum to the same value (i.e., n) for all products that contribute to $y[n]$. The same thing happens in polynomial multiplication because exponents add. In Section 5-3.3.1 on p. 5-3.3.1 we demonstrated a synthetic multiplication tableau for evaluating the convolution of $x[n]$ with $h[n]$. Now we see that this is also a process for multiplying the polynomials $X(z)$ and $H(z)$. The procedure is repeated below for the numerical example of this section.

z	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}	z^{-5}	z^{-6}	z^{-7}
$x[n],\ X(z)$	0	$+1$	-1	$+1$	-1	0	0	0
$h[n],\ H(z)$	1	2	3	4				
$X(z)$	0	$+1$	-1	$+1$	-1	0	0	0
$2z^{-1}X(z)$		0	$+2$	-2	$+2$	-2	0	0
$3z^{-2}X(z)$			0	$+3$	-3	$+3$	-3	0
$4z^{-3}X(z)$				0	$+4$	-4	$+4$	-4
$y[n],\ Y(z)$	0	$+1$	$+1$	$+2$	$+2$	-3	$+1$	-4

In the z-transforms $X(z)$, $H(z)$, and $Y(z)$, the power of z^{-1} is implied by the horizontal position of the coefficient in the tableau. Each row is produced by multiplying the $x[n]$ row by one of the $h[n]$ values and shifting the result right by the implied power of z^{-1}. The final answer is obtained by summing down the columns. The final row is the sequence of values of $y[n] = x[n] * h[n]$ or, equivalently, the coefficients of the polynomial $Y(z)$. ■

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.