Example 7-6: $H(z)$ for Cascade

To give a simple example of this idea, consider a system described by the difference equations

$$w[n] = 3x[n] - x[n-1] \tag{7.22}$$

$$y[n] = 2w[n] - w[n-1] \tag{7.23}$$

which represent a cascade of two first-order systems as in Fig. 7-2. The output $w[n]$ of the first system is the input to the second system, and the overall output is the output of the second system. The intermediate signal $w[n]$ in (7.22) must be computed prior to being used in (7.23). We can combine the two filters into a single difference equation by substituting $w[n]$ from the first system into the second, which gives

$$\begin{aligned} y[n] &= 2w[n] - w[n-1] \\ &= 2(3x[n] - x[n-1]) - (3x[n-1] - x[n-2]) \\ &= 6x[n] - 5x[n-1] + x[n-2] \end{aligned} \tag{7.24}$$

Thus we have proved that the cascade of the two first-order systems is equivalent to a single second-order system. It is important to notice that the difference equation (7.24) defines an algorithm for computing $y[n]$ that is different from the algorithm specified by (7.22) and (7.23) together. However, the above analysis shows that with perfectly accurate computation, the outputs of the two different implementations would be exactly the same. Working out the details of the overall difference equation as we have just done would be extremely tedious if the systems were higher-order. The z-transform simplifies these operations into the multiplication of polynomials. The first-order systems have system functions:

$$H_1(z) = 3 - z^{-1} \qquad \text{and} \qquad H_2(z) = 2 - z^{-1}$$

Therefore, the overall system function is

$$H(z) = (3 - z^{-1})(2 - z^{-1}) = 6 - 5z^{-1} + z^{-2}$$

which matches the difference equation in (7.24). Note that, even in this simple example, the z-domain solution is more straightforward than the n-domain solution. ■

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