EXERCISE 9.4:

Show that the convolution of two exponential signals, $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$, is

$$y(t) = x(t) * h(t) = \frac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

if $a \neq b$

McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458.

SOLUTION

$$x(t) * h(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau$$

Since $u(\tau) = 0$ for $\tau < 0$ and $u(t-\tau) = 0$ for $\tau > t$, the limits on the integral become 0 and t.

$$x(t) * h(t) = \int_{0}^{t} e^{-a\tau} e^{-b(t-\tau)}\, d\tau \qquad \text{if } t \geq 0$$

$$= \int_{0}^{t} e^{-bt} e^{(b-a)\tau}\, d\tau \qquad a \neq b$$

$$= e^{-bt} \left. \frac{e^{(b-a)\tau}}{(b-a)} \right|_{0}^{t}$$

$$= \frac{e^{-bt}}{b-a} \left(e^{(b-a)t} - 1 \right) = \frac{e^{-at} - e^{-bt}}{b-a}$$

If $t < 0$, then $u(\tau) u(t-\tau) = 0$ and the convolution integral is zero. Thus,

$$x(t) * h(t) = \frac{1}{b-a} \left(e^{-at} - e^{-bt} \right) u(t)$$

because $u(t) = 0$ when $t < 0$.

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