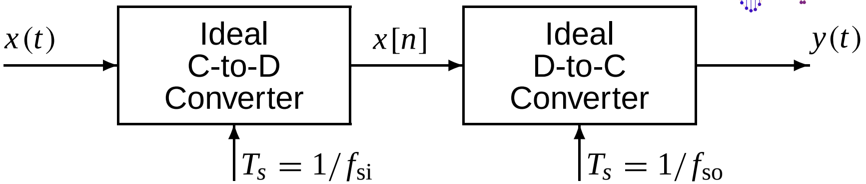
PROBLEM:

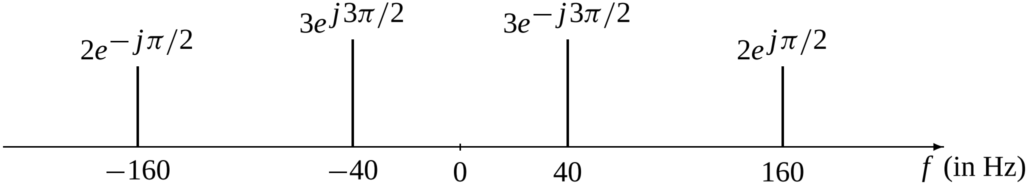


(a) Suppose that the discrete-time signal $x[n]$ is given by the formula

$$x[n] = 10\cos(0.25\pi n - \pi/4)$$

If the sampling rate of the C-to-D converter is $f_{\text{si}} = 2000$ samples/second, many *different* continuous-time signals $x(t) = x_\ell(t)$ could have been inputs to the above system. Determine two such inputs with frequency less than 2000 Hz; i.e., find $x_1(t)$ and $x_2(t)$ such that $x[n] = x_1(nT_{\text{si}}) = x_2(nT_{\text{si}})$ if $T_{\text{si}} = 1/2000$ secs.

(b) Now if the input $x(t)$ is given by the two-sided spectrum representation shown below,



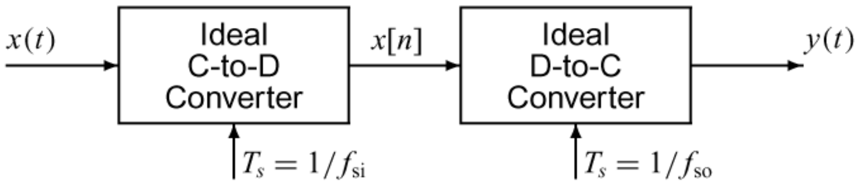
Determine the spectrum for $x[n]$ when $f_{\text{si}} = 120$ samples/sec. Make a $\boxed{\text{plot}}$ for your answer, but label the frequency, amplitude and phase of each spectral component.

(c) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\text{so}} = 120$ Hz.

(d) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\text{so}} = 200$ Hz. In other words, the sampling rates of the two converters are different.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458. © 2003 Pearson Education, Inc.

SOLUTION



(a) Suppose that the discrete-time signal $x[n]$ is given by the formula

$$x[n] = 10\cos(0.25\pi n - \pi/4)$$

If the sampling rate of the C-to-D converter is $f_{\text{si}} = 2000$ samples/second, many *different* continuous-time signals $x(t) = x_\ell(t)$ could have been inputs to the above system. Determine two such inputs with frequency less than 2000 Hz; i.e., find $x_1(t)$ and $x_2(t)$ such that $x[n] = x_1(nT_{\text{si}}) = x_2(nT_{\text{si}})$ if $T_{\text{si}} = 1/2000$.

We can determine $x_1(t)$ using direct substitution as follows:

$$\begin{aligned} x[n] &= x_1(nT_{\text{si}}) \\ 10\cos(0.25\pi n - \pi/4) &= A\cos(2\pi f_1 n T_{\text{si}} + \phi_1) \end{aligned}$$

Using $T_{\text{si}} = 1/2000$, we can solve for $A = 10$, $f_1 = 250$ Hz, and $\phi_1 = -\pi/4$ to give

$$x_1(t) = 10\cos(2\pi \cdot 250t - \pi/4)$$

To determine another input, x_2, that would give the same sampled result, we must find an aliased input which adheres to the following equation

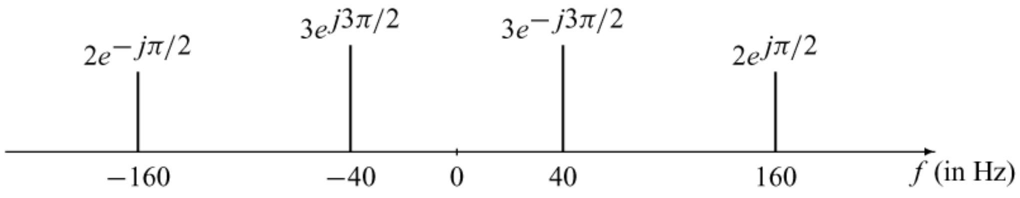
$$x_2(t) = 10\cos(2\pi \cdot (250 + \ell f_{\text{si}})t - \pi/4)$$

where ℓ is an integer. For $f_{\text{si}} = 2000$ samples/second, the only value of ℓ that gives a resulting frequency $f_2 < 2000$ Hz is $\ell = -1$, which gives the following input waveform

$$\begin{aligned} x_2(t) &= 10\cos(2\pi \cdot (250 + (-1) \cdot 2000)t - \pi/4) \\ &= 10\cos(-2\pi \cdot 1750t - \pi/4) \\ &= 10\cos(2\pi \cdot 1750t + \pi/4) \end{aligned}$$

where the final step is a result of the fact that $\cos(\theta) = \cos(-\theta)$ (i.e. cosine is an even function). Note that input $x_2(t)$ results in folding, which causes the phase of the resulting signal to invert (from $\pi/4$ in the input to $-\pi/4$ in the sampled signal).

(b) Now if the input $x(t)$ is given by the two-sided spectrum representation shown below,



Determine the spectrum for $x[n]$ when $f_{\text{si}} = 120$ samples/sec. Make a plot for your answer, but label the frequency, amplitude and phase of each spectral component.

From the spectrum

$$\begin{aligned} x(t) &= 6\cos(2\pi\cdot 40t - 3\pi/2) + 4\cos(2\pi\cdot 160t + \pi/2) \\ &= 6\cos(2\pi\cdot 40t + \pi/2) + 4\cos(2\pi\cdot 160t + \pi/2) \end{aligned}$$

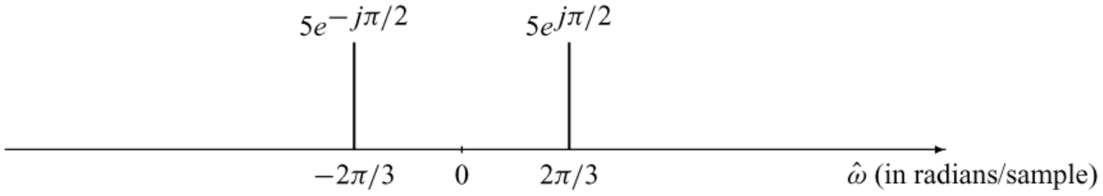
By replacing t with $nT_{\mathrm{si}} = n/120$, we derive

$$x[n] = 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{8\pi}{3}n + \pi/2\right)$$

From this equation, the normalized frequencies of the two terms are $\hat{\omega}_1 = 2\pi/3$ radians/sample and $\hat{\omega}_2 = 8\pi/3$ radians/sample. $\hat{\omega}_1 < \pi$, and thus it does not alias. However, $\hat{\omega}_2 >= \pi$, and thus it does alias. By subtracting 2π from $\hat{\omega}_2$, we find that it aliases to a nomalized frequency of $2\pi/3$ radians/sample. In other words, the two analog frequencies give the same sampled frequency, resulting in them effectively summing, as shown in the following equations:

$$\begin{aligned} x[n] &= 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{8\pi}{3}n + \pi/2\right) \\ &= 6\cos\left(\frac{2\pi}{3}n + \pi/2\right) + 4\cos\left(\frac{2\pi}{3}n + \pi/2\right) \quad = \quad 10\cos\left(\frac{2\pi}{3}n + \pi/2\right) \end{aligned}$$

The resulting spectrum plot is



(c) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\mathrm{so}} = 120$ Hz.

$$f = \frac{\hat{\omega}}{2\pi}\cdot f_{\mathrm{so}} = \frac{1}{3}\cdot 120 = 40\mathrm{Hz}$$

$$x(t) = 10\cos(80\pi t + \pi/2)$$

(d) Using the discrete-time spectrum from part (b), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{\mathrm{so}} = 200$ Hz. In other words, the sampling rates of the two converters are different.

$$f = \frac{\hat{\omega}}{2\pi}\cdot f_{\mathrm{so}} = \frac{1}{3}\cdot 200 \approx 66.67\mathrm{Hz}$$

$$x(t) = 10\cos\left(\frac{400\pi}{3}t + \pi/2\right)$$