PROBLEM:

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) What is the impulse response, $h[n]$, of this system?

(b) Determine the system function $H(z)$ for this system.

(c) Plot the poles and zeros of $H(z)$ in the complex z-plane. *Hint: Remember the N-th roots of unity.*

(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.

(e) Show that your answer in (d) can be expressed in the form

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)} e^{-j2\hat{\omega}}.$$

(f) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above (or plot it using `freqz( )`).

(g) Suppose that the input is

$$x[n] = 5 + 4\cos(0.1\pi n) + 3\cos(0.4\pi n - \pi/4) \qquad \text{for } -\infty < n < \infty$$

Obtain an expression for the output in the form $y[n] = A + B\cos(\hat{\omega}_0 n + \phi_0)$. (In other words, one of the sinusoids is removed by the filter.)

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
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SOLUTION

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) What is the impulse response, $h[n]$, of this system?

$$h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$$

(b) Determine the system function $H(z)$ for this system.

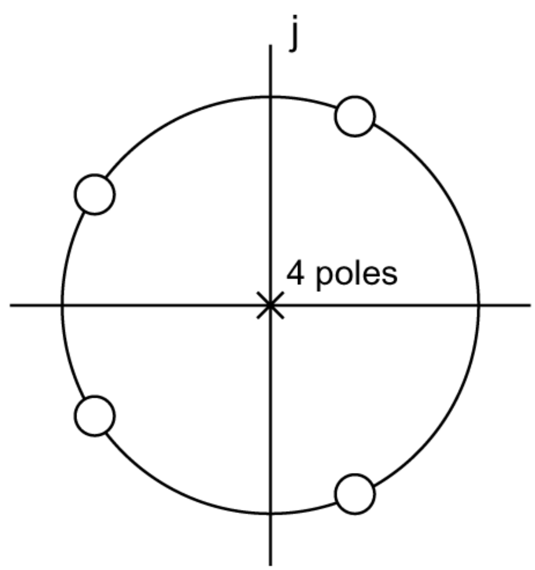
$$H(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right) = \frac{1 - z^{-5}}{5(1 - z^{-1})} = \frac{z^5 - 1}{5z^4(z-1)}$$

(c) Plot the poles and zeros of $H(z)$ in the complex z-plane. *Hint: Remember the N-th roots of unity.*
There is neither a pole or zero at $z = 1$.

Poles: Four at $z = 0$

Zeros: The numerator is $1 - z^{-5}$, therefore, we must solve $z^5 = 1$ and we get $z = e^{j\frac{2\pi n}{5}}$, where $n = 1, 2, 3, 4$.
These zeros are on the unit circle and their angles are multiples of $2\pi/5$ or 72°.

We plot the poles and zeros below.



(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.

$$H(e^{j\hat{\omega}}) = \frac{1}{5}\frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}}$$

(e) Show that your answer in (d) can be expressed in the form $H(e^{j\hat{\omega}}) = \dfrac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)}e^{-j2\hat{\omega}}$.

$$H(e^{j\hat{\omega}}) = \frac{1}{5}\frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}} = \frac{1}{5}e^{-j(5/2)\hat{\omega}}e^{j\hat{\omega}/2}\frac{e^{j(5/2)\hat{\omega}} - e^{-j(5/2)\hat{\omega}}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}} = e^{-j2\hat{\omega}}\frac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)}$$

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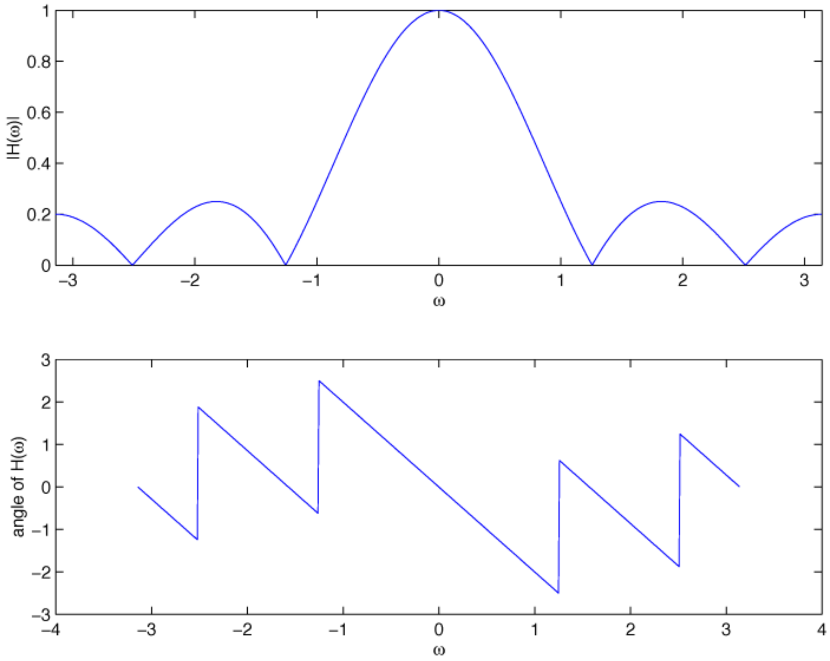
(f) Sketch the frequency response (magnitude and phase) as a function of frequency from the formula above (or plot it using `freqz( )`).



MATLAB Code to make this plot:

```
 ww = [-pi:0.01:pi];
 HH = 0.2 * (sin( 5 * ww / 2) ./ sin( ww / 2) ) .* exp( j*(-2)*ww);
%-- or use [HH,ww] = freqz( ones(1,5)/5, 1, ww);
 subplot(2,1,1)
    plot(w,abs(HH));  axis([-pi pi 0 1]);
 subplot(2,1,2),
    plot(w,angle(HH));
```

(g) Suppose that the input is

$$x[n] = 5 + 4\cos(0.1\pi n) + 3\cos(0.4\pi n - \pi/4) \qquad \text{for } -\infty < n < \infty$$

Obtain an expression for the output in the form $y[n] = A + B\cos(\hat{\omega}_0 n + \phi_0)$. (In other words, one of the sinusoids is removed by the filter.)

For this signal, we need to evaluate the frequency response at three frequencies:

At $\hat{\omega} = 0$, $H(e^{j\hat{\omega}}) = e^{-j0}\dfrac{\sin(0(5/2))}{5\sin(0(1/2))} \to 1.$

At $\hat{\omega} = 0.1\pi$, $H(e^{j\hat{\omega}}) = e^{-j0.2\pi}\dfrac{\sin(0.25\pi)}{5\sin(0.05\pi)} = 0.904e^{-j0.2\pi}.$

At $\hat{\omega} = 0.4\pi$, $H(e^{j\hat{\omega}}) = e^{-j0.8\pi}\dfrac{\sin(\pi)}{5\sin(\pi/5)} = 0.$

Therefore the solution is

$$x[n] = 5(1) + 4(0.904)\cos(0.1\pi n - 0.2\pi) + 3(0)\cos(0.4\pi n - \pi/4) = 5 + 3.616\cos(0.1\pi n - 0.2\pi)$$