PROBLEM:

The diagram in Figure 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system. In Figure 1, assume that both systems are 3-point moving averagers; i.e.,

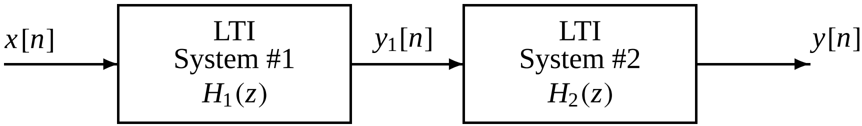

Figure 1: Cascade connection of two LTI systems.

$$y_1[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) \quad \text{and} \quad y[n] = \frac{1}{3}(y_1[n] + y_1[n-1] + y_1[n-2]).$$

(a) Determine the system function $H(z) = H_1(z)H_2(z)$ for the overall system.

(b) Plot the poles and zeros of $H(z)$ in the z-plane.

(c) Use multiplication of z-transform polynomials to determine the impulse response $h[n]$ of the overall system in Figure 1.

(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of the overall cascade system.

(e) Use your result from (d) as an aid in sketching the frequency response (magnitude and phase) functions of the overall cascade system for $-\pi \leq \hat{\omega} \leq \pi$.

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SOLUTION

The diagram in Figure 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system. In Figure 1,

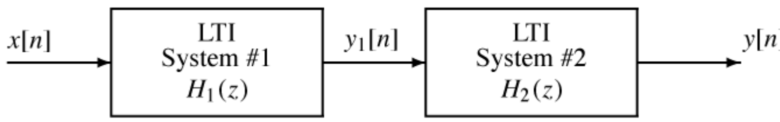


Figure 1: Cascade connection of two LTI systems.

assume that both systems are 3-point moving averagers; i.e.,

$$y_1[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) \quad \text{and} \quad y[n] = \frac{1}{3}(y_1[n] + y_1[n-1] + y_1[n-2]).$$

(a) Determine the system function $H(z) = H_1(z)H_2(z)$ for the overall system.

$$H_1(z) = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right)$$

$$H_2(z) = \frac{1}{3}\left(1 + z^{-1} + z^{-2}\right)$$

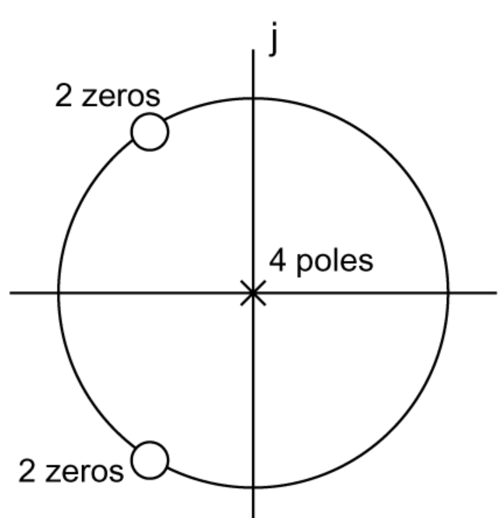
$$H(z) = H_1(z)H_2(z) = \frac{1}{9}\left(1 + z^{-1} + z^{-2}\right)^2$$

$$= \frac{1}{9}\left(1 + z^{-2} + z^{-4} + 2z^{-1} + 2z^{-2} + 2z^{-3}\right) = \frac{1}{9}\left(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}\right)$$

(b) Plot the poles and zeros of $H(z)$ in the z-plane.

$$H(z) = H_1(z)H_2(z) = \frac{1}{9}\left(1 + z^{-1} + z^{-2}\right)^2 = \frac{1}{9}\left(\frac{1 - z^{-3}}{1 - z^{-1}}\right)^2 = \frac{(z^3 - 1)^2}{9z^4(z-1)^2}$$

Thus, we have four poles at $z = 0$, and double zeros at both $z = e^{j2\pi/3}$ and $z = e^{j4\pi/3}$. There is no pole or zero at $z = 1$, because $z = 1$ is a root of both the numerator and the denominator. We plot the poles and zeros below.



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(c) Use multiplication of z-transform polynomials to determine the impulse response $h[n]$ of the overall system in Figure 1.

This was already done in part (a). The impulse response is the inverse z-transform:

$$h[n] = \frac{1}{9}\left(\delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]\right)$$

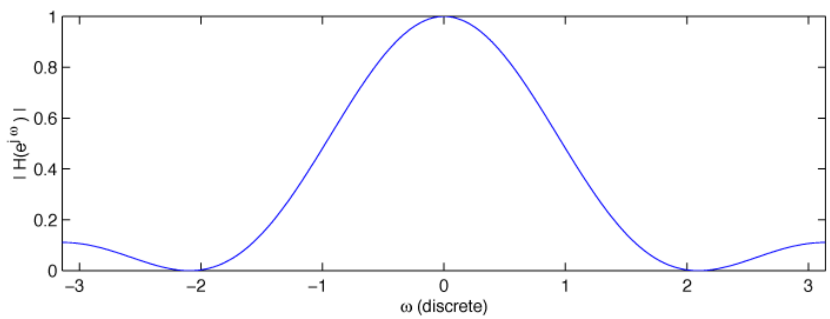
(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of the overall cascade system.

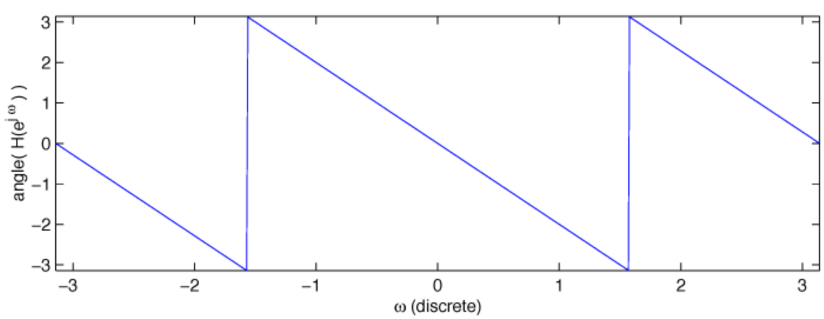
We know from earlier results that the frequency response of the 3-pt averager is: $\frac{1}{3}e^{-j\hat{\omega}}\frac{\sin((3/2)\hat{\omega})}{\sin(\hat{\omega}/2)}$

Therefore, since $H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$, we get

$$H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}}\frac{\sin^2((3/2)\hat{\omega})}{9\sin^2(\hat{\omega}/2)}$$

(e) Use your result from (d) as an aid in sketching the frequency response (magnitude and phase) functions of the overall cascade system for $-\pi \leq \hat{\omega} \leq \pi$.





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