PROBLEM:

The system function of a linear time-invariant filter is given by the formula

$$H(z) = (1 - z^{-1})(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1})(1 - 0.5e^{j\pi/3}z^{-1})(1 - 0.5e^{-j\pi/3}z^{-1})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. Make sure that all the filter coefficients $\{b_k\}$ in your difference equation are purely real.

(b) What is the output if the input is $x[n] = \delta[n]$?

(c) Use multiplication of z-transform polynomials to find the output when the input is

$$x[n] = \delta[n-2] + 2\delta[n-4] - \delta[n-5].$$

(d) Plot the poles and zeros of $H(z)$ in the z-plane.

(e) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.

(f) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of frequency $\hat{\omega}$ will the output signal be zero for all n (i.e., $y[n] = 0$) ? Find all possible frequencies in the range $-\pi \leq \hat{\omega} \leq \pi$. *Hint: Take a look at the locations of the zeros of $H(z)$ as plotted in part (d).*

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SOLUTION

The system function of a linear time-invariant filter is given by the formula

$$H(z) = (1 - z^{-1})(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1})(1 - 0.5e^{j\pi/3}z^{-1})(1 - 0.5e^{-j\pi/3}z^{-1})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. Make sure that all the filter coefficients $\{b_k\}$ in your difference equation are purely real.
A few simplifications:

$$(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1}) = 1 - \left(e^{j\pi/2} + e^{-j\pi/2}\right) z^{-1} + e^{j\pi/2}e^{-j\pi/2}z^{-2} = 1 + z^{-2}$$

$$(1 - 0.5e^{j\pi/3}z^{-1})(1 - 0.5e^{-j\pi/3}z^{-1}) = 1 - 0.5\left(e^{j\pi/3} + e^{-j\pi/3}\right) z^{-1} + (0.5)^2 e^{j\pi/3}e^{-j\pi/3}z^{-2} = 1 - 0.5z^{-1} + 0.25z^{-2}$$

Therefore,

$$\begin{aligned} H(z) &= (1 - z^{-1})(1 + z^{-2})(1 - 0.5z^{-1} + 0.25z^{-2}) \\ &= (1 - z^{-1} + z^{-2} - z^{-3})(1 - 0.5z^{-1} + 0.25z^{-2}) \\ &= 1 - 1.5z^{-1} + 1.75z^{-2} - 1.75z^{-3} + 0.75z^{-4} - 0.25z^{-5} \end{aligned}$$

The difference equation is

$$y[n] = x[n] - 1.5x[n-1] + 1.75x[n-2] - 1.75x[n-3] + 0.75x[n-4] - 0.25x[n-5]$$

(b) What is the output if the input is $x[n] = \delta[n]$?
The output is the impulse response:

$$h[n] = \delta[n] - 1.5\delta[n-1] + 1.75\delta[n-2] - 1.75\delta[n-3] + 0.75\delta[n-4] - 0.25\delta[n-5]$$

(c) Use multiplication of z-transform polynomials to find the output when the input is

$$x[n] = \delta[n-2] + 2\delta[n-4] - \delta[n-5].$$

First of all, we need to get $X(z)$, the z-transform of $x[n]$:

$$X(z) = z^{-2} + 2z^{-4}z^{-5}$$

Then we do the polynomial multiplication of $H(z)$ and $X(z)$:

$$\begin{aligned} X(z)H(z) &= (z^{-2} + 2z^{-4}z^{-5})(1 - 1.5z^{-1} + 1.75z^{-2} - 1.75z^{-3} + 0.75z^{-4} - 0.25z^{-5}) \\ &= z^{-2} - 1.5z^{-3} + 3.75z^{-4} - 5.75z^{-5} + 5.75z^{-6} - 5.5z^{-7} + 3.25z^{-8} - 1.25z^{-9} + 0.25z^{-10} \end{aligned}$$

We can also do "synthetic multiplication" to form the following table of terms. We know from the exponents of z that the minimum delayed response is at $n = 2$, and the maximum delayed response is at $n = 10$.

Input	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$\delta[n-2]$	0	0	1	-1.5	1.75	-1.75	0.75	-0.25	0	0	0
$2\delta[n-4]$	0	0	0	0	2	-3	3.5	-3.5	1.5	-0.5	0
$-\delta[n-5]$	0	0	0	0	0	-1	1.5	-1.75	1.75	-0.75	0.25
Sum	0	0	1	-1.5	3.75	-5.75	5.75	-5.5	3.25	-1.25	0.25

$$\begin{aligned} y[n] = \delta[n-2] - 1.5\delta[n-3] + 3.75\delta[n-4] - 5.75\delta[n-5] + 5.75\delta[n-6] \\ -5.5\delta[n-7] + 3.25\delta[n-8] - 1.25\delta[n-9] + 0.25\delta[n-10] \end{aligned}$$

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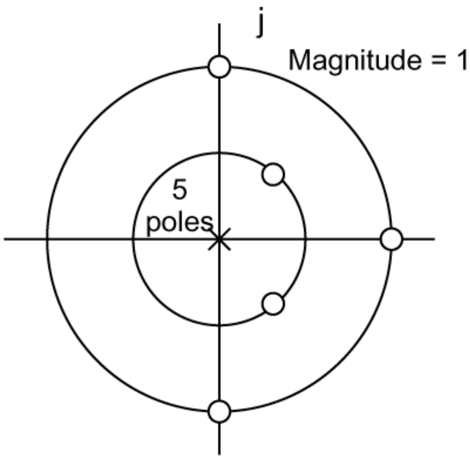
(d) Plot the poles and zeros of $H(z)$ in the z-plane.



(e) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of this system.
We just replace z with $e^{j\omega}$, we get:

$$H(e^{j\hat{\omega}}) = 1 - 1.5e^{-j\hat{\omega}} + 1.75e^{-j2\hat{\omega}} - 1.75e^{-j3\hat{\omega}} + 0.75e^{-j4\hat{\omega}} - 0.25e^{-j5\hat{\omega}}$$

but it is not possible to do much simplification because the impulse response is not symmetric. Here is one way to simplify:

$$\begin{aligned} H(e^{j\hat{\omega}}) &= (1 - e^{-j\hat{\omega}})(1 + e^{-j2\hat{\omega}})(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\ &= (1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}})(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\ &= \frac{1 - e^{-j4\hat{\omega}}}{1 - (-e^{-j\hat{\omega}})}(1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \\ &= e^{-j1.5\hat{\omega}} e^{j\pi/2} \left(\frac{\sin(2\hat{\omega})}{\cos(\hat{\omega}/2)} \right) (1 - 0.5e^{-j\hat{\omega}} + 0.25e^{-j2\hat{\omega}}) \end{aligned}$$

(f) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of frequency $\hat{\omega}$ will the output signal be zero for all n (i.e., $y[n] = 0$) ? Find all possible frequencies in the range $-\pi \le \hat{\omega} \le \pi$. *Hint: Take a look at the locations of the zeros of $H(z)$ as plotted in part (d).*
We can use the factored form of $H(z)$, or the zeros of $H(z)$, to find the zeros that are *on the unit circle*. These are $z = 1$, $z = e^{j\pi/2}$ and $z = e^{-j\pi/2}$. These three zeros all have a magnitude equal to 1. Using the relationship between the z and $\hat{\omega}$ domains, $z = e^{j\hat{\omega}}$, we see that the three frequencies are $\hat{\omega} = 0$, $\hat{\omega} = \pi/2$ and $\hat{\omega} = -\pi/2$. The output for three different sinusoids will be zero. For example, when the frequency is $\hat{\omega} = \pi/2$

$$x[n] = e^{j(\pi/2)n} \quad \rightarrow \quad y[n] = H(e^{j\pi/2})e^{j(\pi/2)n} = 0$$

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