PROBLEM:

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) Find the output $y_1[n]$ when the input is

$$x_1[n] = 10\delta[n-50].$$

(b) Find the output $y_2[n]$ when the input is

$$x_2[n] = \begin{cases} 1 & 0 \le n \le 10 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the output $y_3[n]$ when the input is

$$x_3[n] = 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty.$$

(d) Use the concept of linearity to find the output $y_4[n]$ when the input is

$$x_4[n] = 10\delta[n-50] + 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty$$

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SOLUTION

A linear time-invariant filter is described by the difference equation

$$y[n] = \frac{1}{5}\{x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\} = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

(a) Find the output $y_1[n]$ when the input is

$$x_1[n] = 10\delta[n-50]$$

First, find $h[n]$, the finite impulse response of the filter:

$$h[n] = \frac{1}{5}\sum_{k=0}^{4} \delta[n-k]$$

Then convolve $h[n]$ with the input:

$$\begin{aligned} y_1[n] &= h[n] * 10\delta[n-50] \\ &= 10h[n-50] \\ &= 2 \cdot \sum_{k=50}^{54} \delta[n-k] \end{aligned}$$

(b) Find the output $y_2[n]$ when the input is

$$x_2[n] = \begin{cases} 1 & 0 \leq n \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the sequences for $x_2[n]$ and $h[n]$:

$$\begin{aligned} \{x_2[n]\} &= \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\} \\ \{h[n]\} &= \{1/5, 1/5, 1/5, 1/5, 1/5\} \end{aligned}$$

Convolve the two sequences:

$h[0] * x_2[n]$	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5				
$h[1] * x_2[n-1]$		1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5			
$h[2] * x_2[n-2]$			1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5		
$h[3] * x_2[n-3]$				1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	
$h[4] * x_2[n-4]$					1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5
$h[n] * x[n]$	1/5	2/5	3/5	4/5	1	1	1	1	1	1	1	4/5	3/5	2/5	1/5

The values of the output signal are given below, starting at $n = 0$ and ending at $n = 14$.

$$\{y_2[n]\} = \{1/5, 2/5, 3/5, 4/5, 1, 1, 1, 1, 1, 1, 1, 4/5, 3/5, 2/5, 1/5\}$$

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(c) Find the output $y_3[n]$ when the input is

$$x_3[n] = 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty$$

Step 1 Find $H(e^{j\hat{\omega}})$, the frequency response of the 5-point running averager using the Dirichlet function:

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega}/2)}{5\sin(\hat{\omega}/2)} e^{-j2\hat{\omega}}$$

Step 2 Evaluate the frequency response at the frequency of each of the cosine terms.

$$\begin{aligned} H(e^{j0.1\pi}) &= \frac{\sin(0.5\pi/2)}{5\sin(0.1\pi/2)} e^{-j0.2\pi} = \frac{\sin(\pi/4)}{5\sin(\pi/20)} e^{-j\pi/5} \\ &\approx 0.904 e^{-j\pi/5} \end{aligned}$$

$$\begin{aligned} H(e^{j0.4\pi}) &= \frac{\sin(5(0.4\pi)/2)}{5\sin(0.4\pi/2)} e^{-j2(0.4\pi)} = \frac{\sin(\pi)}{5\sin(\pi/5)} e^{-j4\pi/5} \\ &= 0 \end{aligned}$$

Step 3 Find $y[n]$ by multiplying the amplitudes and adding the phases for each pair of cosine / frequency-response terms.

$$\begin{aligned} y_3[n] &\approx (4 \cdot 0.904)\cos(0.1\pi n + \pi/2 - \pi/5) + (3 \cdot 0)\cos(0.4\pi n - \pi) \\ &\approx 3.616\cos(0.1\pi n + 0.3\pi) \end{aligned}$$

(d) Use the concept of linearity to find the output $y_4[n]$ when the input is

$$x_4[n] = 10\delta[n-50] + 4\cos(0.1\pi n + \pi/2) + 3\cos(0.4\pi n - \pi) \qquad \text{for } -\infty < n < \infty$$

From the previous parts, we see that $x_4[n] = x_1[n] + x_3[n]$. Thus, using linearity, we know that $y_4[n] = y_1[n] + y_3[n]$, resulting in

$$y_4[n] \approx 2 \cdot \sum_{k=50}^{54} \delta[n-k] + 3.616\cos(0.1\pi n + 0.3\pi)$$

The approximation sign is used because the value 3.616 is approximate.