PROBLEM:

A linear time-invariant system is described by the difference equation

$$y[n] = 0.8y[n-1] + x[n] + x[n-1].$$

(a) Suppose that the input is the unit step sequence, i.e., $x[n] = u[n]$. The output signal will be infinitely long. Determine by iterating the difference equation, the output of this system for the range $0 \leq n \leq 10$.

(b) Suppose that the input is the pulse sequence

$$x[n] = u[n] - u[n-6] = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Once again, the output signal will be infinitely long. Determine the output of this system for $0 \leq n \leq 10$. *You can do this either by iterating the difference equation as you did in part (a), or you may apply linearity and time invariance to the answer that you computed in part (a). Do the problem either or both ways.*

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458.

SOLUTION

A linear time-invariant system is described by the difference equation

$$y[n] = 0.8y[n-1] + x[n] + x[n-1].$$

(a) Suppose that the input is the unit step sequence, i.e., $x[n] = u[n]$. The output signal will be infinitely long. Determine by iterating the difference equation, the output of this system for the range $0 \le n \le 10$.

Recall that the unit step signal is zero for $n < 0$ and one for $n \ge 0$. The *initial rest condition* is our standard assumption for initial conditions when the IIR filter is to be a linear time-invariant system, so we have $y[n] = 0$ for $n < 0$, and then we can calculate the values one at time:

$$\begin{aligned}
y[0] &= (0.8)(0) + (1) + (0) = 1 \\
y[1] &= (0.8)(1) + (1) + (1) = 2.8 \\
y[2] &= (0.8)(2.8) + (1) + (1) = 4.24 \\
y[3] &= (0.8)(4.24) + (1) + (1) = 5.392 \\
y[4] &= (0.8)(5.392) + (1) + (1) \approx 6.314 \\
y[5] &\approx (0.8)(6.314) + (1) + (1) \approx 7.051 \\
y[6] &\approx (0.8)(7.051) + (1) + (1) \approx 7.641 \\
y[7] &\approx (0.8)(7.641) + (1) + (1) \approx 8.113 \\
y[8] &\approx (0.8)(8.113) + (1) + (1) \approx 8.490 \\
y[9] &\approx (0.8)(8.490) + (1) + (1) \approx 8.792 \\
y[10] &\approx (0.8)(8.792) + (1) + (1) \approx 9.034
\end{aligned}$$

• Notice that the general solution for $n \ge 0$ is $y[n] = 10 - 9(0.8)^n$, which contains a term that is "pole to the n." Also $y[n]$ converges to 10 as $n \to \infty$.

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(b) Suppose that the input is the pulse sequence

$$x[n] = u[n] - u[n-6] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

Once again, the output signal will be infinitely long. Determine the output of this system for $0 \le n \le 10$.

Iteration Method

If we define $y_b[n]$ to be the solution for this problem, and assume that $y_b[n] = 0$ for $n < 0$:

$$\begin{aligned}
y_b[0] &= (0.8)(0) + (1) + (0) = 1 \\
y_b[1] &= (0.8)(1) + (1) + (1) = 2.8 \\
y_b[2] &= (0.8)(2.8) + (1) + (1) = 4.24 \\
y_b[3] &= (0.8)(4.24) + (1) + (1) = 5.392 \\
y_b[4] &= (0.8)(5.392) + (1) + (1) \approx 6.314 \\
y_b[5] &\approx (0.8)(6.314) + (1) + (1) \approx 7.051 \\
y_b[6] &\approx (0.8)(7.051) + (0) + (1) \approx 6.641 \\
y_b[7] &\approx (0.8)(6.641) + (0) + (0) \approx 5.313 \\
y_b[8] &\approx (0.8)(5.313) + (0) + (0) \approx 4.250 \\
y_b[9] &\approx (0.8)(4.250) + (0) + (0) \approx 3.400 \\
y_b[10] &\approx (0.8)(3.400) + (0) + (0) \approx 2.720
\end{aligned}$$

- Notice that the general solution for $n \ge 6$ is $y_b[n] = 6.641(0.8)^{n-6}$, which is "pole to the n."

Applying Linearity and Time Invariance

If we define $y_b[n]$ to be the solution for this problem and use $y[n]$ from the previous part:

$$y_b[n] = y[n] - y[n-6]$$

resulting in the following solutions

$$\begin{aligned}
y_b[0] &= y[0] - y[-6] = 1 - 0 = 1 \\
y_b[1] &= y[1] - y[-5] = 2.8 - 0 = 2.8 \\
y_b[2] &= y[2] - y[-4] = 4.24 - 0 = 4.24 \\
y_b[3] &= y[3] - y[-3] = 5.392 - 0 = 5.392 \\
y_b[4] &= y[4] - y[-2] \approx 6.314 - 0 \approx 6.314 \\
y_b[5] &= y[5] - y[-1] \approx 7.051 - 0 \approx 7.051 \\
y_b[6] &= y[6] - y[0] \approx 7.641 - 1 \approx 6.641 \\
y_b[7] &= y[7] - y[1] \approx 8.113 - 2.8 \approx 5.313 \\
y_b[8] &= y[8] - y[2] \approx 8.490 - 4.24 \approx 4.250 \\
y_b[9] &= y[9] - y[3] \approx 8.792 - 5.392 \approx 3.400 \\
y_b[10] &= y[9] - y[3] \approx 9.034 - 6.314 \approx 2.720
\end{aligned}$$

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