PROBLEM:

A continuous-time system is defined by the input/output relation

$$y(t) = \int_{t-2}^{t-1} x(\tau)d\tau.$$

(a) Show that this is a linear time-invariant system.

(b) Determine the impulse response, $h(t)$, of this system. Plot it.

(c) Is this a stable system? Is it a causal system?

(d) Use convolution to determine the output of the system when the input is

$$x(t) = u(t) - u(t-1) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Plot your answer.

(e) Use convolution to determine the output when the input is

$$x(t) = e^{-2t}u(t).$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.

SOLUTION

A continuous-time system is defined by the input/output relation

$$y(t) = \int_{t-2}^{t-1} x(\tau)d\tau.$$

(a) Show that this is a linear time-invariant system.

Solution: First, we test linearity. If we input a signal $x(t) = a_1x_1(t) + a_2x_2(t)$, then the output response is

$$\begin{aligned} y(t) &= \int_{t-2}^{t-1} (a_1x_1(\tau) + a_2x_2(\tau))d\tau \\ &= a_1\left(\int_{t-2}^{t-1} x_1(\tau)d\tau\right) + a_2\left(\int_{t-2}^{t-1} x_2(\tau)d\tau\right) \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

where $y_1(t)$, $y_2(t)$ are the responses to $x_1(t)$, $x_2(t)$.

In order to test time-invariance, we input a signal $x(t) = x_2(t - t_{\mathrm{d}})$, then the output response is

$$\begin{aligned} y(t) &= \int_{t-2}^{t-1} x_2(\tau - t_d)d\tau \\ &= \int_{t-2}^{t-1} x_2(\tau - t_d)d\tau \\ &= \int_{t-2-t_d}^{t-1-t_d} x_2(\lambda)d\lambda \\ &= \int_{(t-t_d)-2}^{(t-t_d)-1} x_2(\lambda)d\lambda \;=\; y_2(t - t_d) \end{aligned}$$

where we replaced $(\tau - t_d)$ with λ, and $d\tau$ with $d\lambda$ in step #3.

(b) Determine the impulse response, $h(t)$, of this system. Plot it.

Solution:

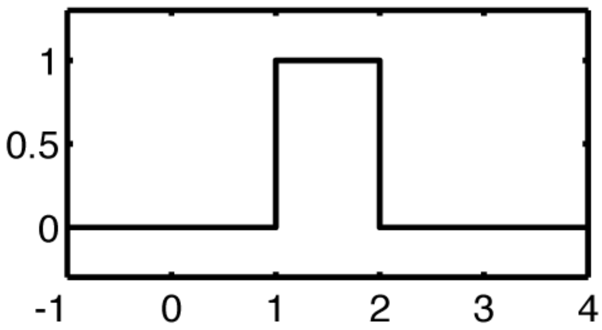
$$\begin{aligned} h(t) &= y(t) = \int_{t-2}^{t-1} x(\tau)d\tau \\ &= y(t) = \int_{t-2}^{t-1} \delta(\tau)d\tau \end{aligned}$$

The integral of $\delta(\tau)$ will be non-zero if the limits of integration include the location of the impulse at $\tau = 0$. This is true if $(t-2) < 0$ and $(t-1) > 0$, or equivalently, the impulse response is equal to one when $1 < t < 2$; it is zero everywhere else. Therefore, the impulse response can be written in the following clever fashion.

$$h(t) = u(t-1) - u(t-2)$$

The difference of the two unit-step signals is actually a pulse of length one. We show the plot below.

McClellan, Schafer, and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.



(c) Is this a stable system? Is it a causal system?

Solution: This system is stable since the total integral of $|h(t)|$ is finite.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{1}^{2} |h(t)|dt = 1 < \infty$$

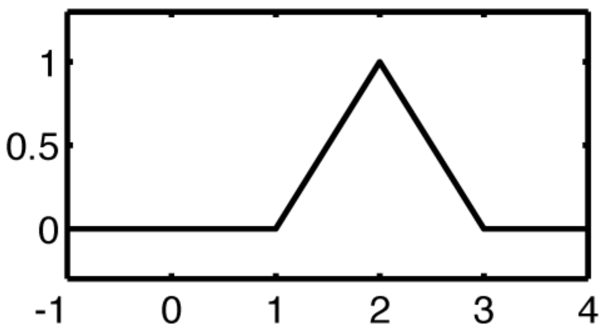
The system is causal because the impulse response does not start before $t = 0$. Thus the output signal $y(t)$ at time t only requires values of the input in the time reange $t-1$ to $t-2$ which is prior to time t.

(d) Use convolution to determine the output of the system when the input is

$$x(t) = u(t) - u(t-1) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Plot your answer.

Solution: This problem is the convolution of two boxes, of unit height and unit width. The solution is a triangle function of width 2 with its peak value equal to 1. In this case, since the box for $h(t)$ is delayed one second in time, the solution is delayed by one second in time. We plot this solution below.



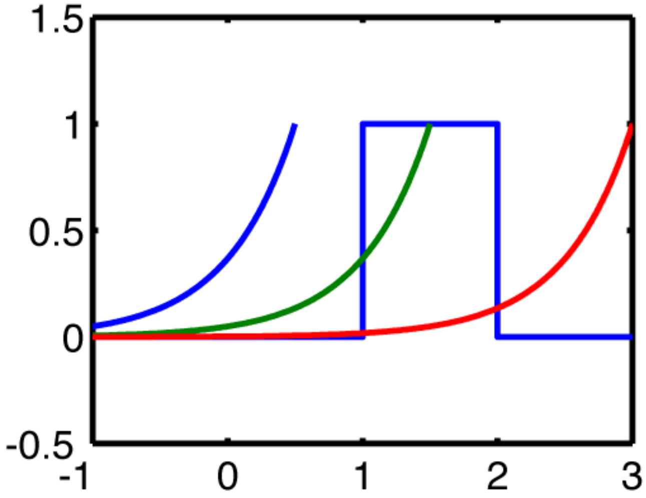
(e) Use convolution to determine the output when the input is

$$x(t) = e^{-2t}u(t).$$

Solution: We define convolution as

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

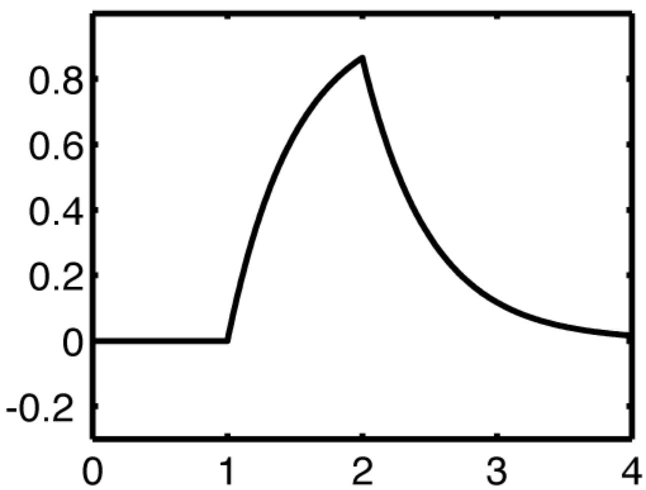
McClellan, Schafer, and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458. © 2003 Pearson Education, Inc.



We have three cases, illustrated by the figure above. We choose to flip the input, and not $h(t)$. In the first case, $t < 1$, there is no overlap between between $h(\tau)$ and $x(\tau - t)$. In the second and third cases, $1 < t$, there is overlap. In the third case, the overlap ends at $\tau = 2$. We generated the above plot using the following MATLAB commands:

```
xx1 = [0:0.01:4];
subplot(2,2,1)
plot([-1 1 1 2 2 3],[0 0 1 1 0 0],'b-')
hold on
plot(xx1-3.5,exp(2*(xx1 - 4)),xx1-2.5,exp(2*(xx1 - 4)),xx1-1,exp(2*(xx1 - 4)));
hold off
axis([-1 3 -0.5 1.5]);
```

The solution is $y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{2}\left(1 - e^{-2(t-1)}\right) & 1 < t < 2 \\ \frac{1}{2}e^{-2(t-2)}\left(1 - e^{-2}\right) & t > 2 \end{cases}$



We generated this plot with the following MATLAB commands:

McClellan, Schafer, and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.

```
xx = [0:0.01:1];
xx2 = [0:0.01:2];
plot([0 xx+1 xx2+2],[0 (1 - exp(-2*xx)) exp(-2*xx2)*(1 - exp(-2) )]);
```

Here is the integral for case #2 which applies when $1 < t < 2$:

$$\begin{aligned} \int_1^t e^{-2(t-\tau)} d\tau &= \tfrac{1}{2} e^{-2t}\, e^{2\tau}\Big|_1^t \\ &= \tfrac{1}{2} e^{-2t} \left(e^{2t} - 1\right) = \tfrac{1}{2}\left(1 - e^{-2(t-1)}\right) \end{aligned}$$

Here is the integral for case #3 which applies when $2 < t$:

$$\begin{aligned} \int_1^2 e^{-2(t-\tau)} d\tau &= \tfrac{1}{2} e^{-2t}\, e^{2\tau}\Big|_1^2 \\ &= \tfrac{1}{2} e^{-2t} \left(e^{4} - e^{2}\right) = \tfrac{1}{2}\left(e^{-2(t-2)} - e^{-2(t-1)}\right) \end{aligned}$$