PROBLEM:

Here is a Fourier transform that will be useful in the lab project for Labs #9 and #10.

(a) Define the finite duration signal $x(t)$ to be one half cycle of a sine wave:

$$x(t) = \begin{cases} \sin(\pi t) & \text{for } 0 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Make a plot of $x(t)$ over the range $-3 \leq t \leq 3$.

(b) Determine $X(j\omega)$, the Fourier transform of $x(t)$. One approach is to use Euler's formula to break the integral down into integrating two complex exponentials.

(c) Plot the magnitude and phase of $X(j\omega)$ from the previous part. You will probably have to use MATLAB to make these plots from the formula that you derive.

(d) Define a new signal

$$q(t) = \begin{cases} \cos(100\pi t) & \text{for } |t| \leq 0.005 \\ 0 & \text{elsewhere} \end{cases}$$

Use the scaling and shifting properties of the Fourier transform to write the formula for $Q(j\omega)$. Hint: if you write $q(t)$ in terms of $x(t)$ as $q(t) = x(\alpha t - \beta)$, then you can apply the scaling and shifting properties. Which order should you use? Scale first and then shift, or vice versa?

(e) Prove that the $Q(j\omega)$ from the previous part is a purely real function; no imaginary part.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
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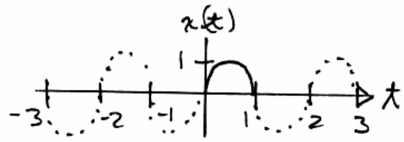
SOLUTION

a) $X(j\omega) = \int_0^1 \sin(\pi t)\, e^{-j\omega t}\, dt$



b) $= \frac{1}{2j}\int_0^1 \left(e^{j\pi t} - e^{-j\pi t}\right) e^{-j\omega t}\, dt$

From Euler: $\sin(\omega t) = \frac{e^{+j\omega t} - e^{-j\omega t}}{2j}$

$$= \frac{1}{2\cancel{j}}\left[\frac{e^{-jt(\omega-\pi)}}{-\cancel{j}(\omega-\pi)}\Bigg|_0^1 - \frac{e^{-jt(\omega+\pi)}}{-\cancel{j}(\omega+\pi)}\Bigg|_0^1\right]$$

$$= \frac{1}{2}\left[\left(\frac{\omega+\pi}{\omega-\pi}\right)\cdot\frac{e^{-j(\omega-\pi)}-1}{(\omega-\pi)} - \frac{(\omega-\pi)}{(\omega-\pi)}\cdot\frac{e^{-j(\omega+\pi)}-1}{(\omega+\pi)}\right]$$

$$= \frac{1}{2(\omega^2-\pi^2)}\left[\omega e^{-j(\omega-\pi)} - \cancel{\omega} + \pi e^{-j(\omega-\pi)} - \pi - \left(\omega e^{-j(\omega+\pi)} - \cancel{\omega} - \pi e^{-j(\omega+\pi)} + \pi\right.\right.$$

Note: $e^{-j(\omega-\pi)} = e^{-j\omega}\left(e^{-j\pi} = -1\right) = -e^{-j\omega}$

and

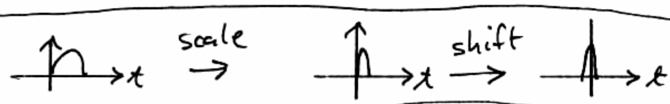
$\cancel{e^{-j}}\; e^{-j(\omega+\pi)} = e^{-j\omega}\left(e^{j\pi} = -1\right) = -e^{-j\omega}$

$$= -\frac{2\pi}{2}\,\frac{(1+e^{-j\omega})}{(\omega^2-\pi^2)} = \frac{2\pi\,(e^{-j\omega}+1)}{\pi^2-\omega^2} = \frac{2\pi}{(\pi^2-\omega^2)2}\, e^{-j\frac{\omega}{2}}\left(e^{-j\frac{\omega}{2}} + e^{j\frac{\omega}{2}}\right)$$

$$= \frac{2\pi\cos(\omega/2)}{\pi^2-\omega^2}\cdot e^{-j\frac{\omega}{2}} = |X(\omega)|\cdot\angle X(\omega)$$

c) See MATLAB plots

d) Scale then shift:



Scaled:

$$\sin(100t)\Big|_0^{0.01} \leftrightarrow \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}\, e^{-j\frac{\omega}{200}}$$

Use properties:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(\omega)$$

Shifted:

$$\cos(100t)\Big|_{-0.005}^{0.005} \leftrightarrow \frac{e^{j\omega\frac{1}{200}}}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}\, e^{-j\frac{\omega}{200}} = \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2}$$

e) Note that exp() terms cancel so that:

$$\cos(100t)\left[u\left(t+\tfrac{1}{200}\right) - u\left(t-\tfrac{1}{200}\right)\right] \leftrightarrow \frac{1}{100}\,\frac{2\pi\cos\left(\frac{\omega}{200}\right)}{\pi^2-\left(\frac{\omega}{100}\right)^2} = \text{All Real}$$

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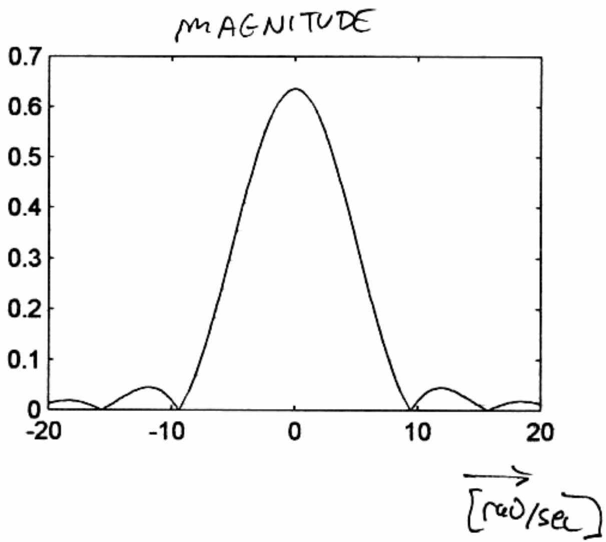
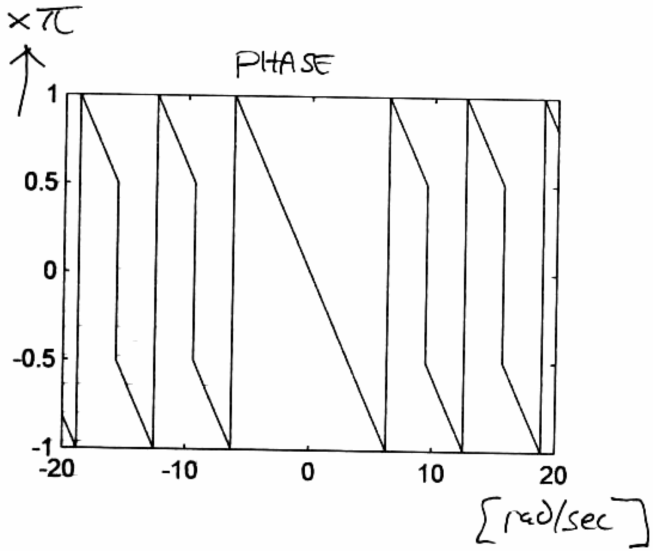
PLOTS





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