PROBLEM:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

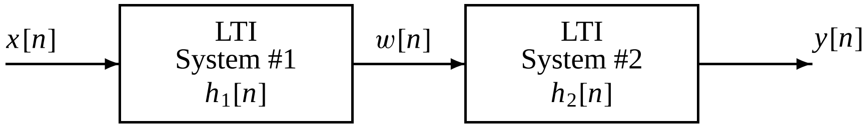

Figure 1: Cascade connection of two LTI systems.

(a) Suppose that LTI System #1 is described by the difference equation

$$w[n] = x[n] - \alpha x[n-1].$$

Determine the impulse response $h_1[n]$ of the first system.

(b) The LTI System #2 is described by the impulse response

$$h_2[n] = \alpha^n (u[n] - u[n-L]) = \sum_{k=0}^{L-1} \alpha^k \delta[n-k] = \begin{cases} \alpha^n & n = 0, 1, \ldots, L-1 \\ 0 & \text{otherwise.} \end{cases}$$

For the special case of $L = 5$, use convolution to show that the impulse response sequence of the overall cascade system is

$$h[n] = h_1[n] * h_2[n] = \delta[n] - \alpha^5 \delta[n-5].$$

(c) Generalize your result in part (b) for the general case of L any integer value.

(d) Obtain a single difference equation that relates $y[n]$ to $x[n]$ in Fig. 1.

(e) Assuming that $0 < \alpha < 1$, how would you choose L so that $y[n] = x[n]$ in Figure 1; i.e., how would you choose L so that the second system "undoes" the effect of the first system?

You will use the results of this problem in Lab #6.

McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458. © 2003 Pearson Education, Inc.

SOLUTION

a) $h_1[n] = \delta[n] - \alpha\,\delta[n-1]$

(by definition, the impulse response is the system's output when the input is $\delta[n]$).
In the difference equation, set $x[n] = \delta[n]$

(b) Use numerical convolution

$$
\begin{array}{cccccc}
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \\
1 & -\alpha & & & & \\
\hline
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \\
 & -\alpha & -\alpha^2 & -\alpha^3 & -\alpha^4 & -\alpha^5 \\
\hline
1 & 0 & 0 & 0 & 0 & -\alpha^5
\end{array}
$$

$n=5$ column

Thus, $h[n] = \delta[n] - \alpha^5\delta[n-5]$

(c) In part (b) $L=5$, so we suspect that

$h[n] = \delta[n] - \alpha^L\delta[n-L]$ for any L

Here is a general derivation:

$$
\begin{aligned}
h_1[n] * h_2[n] &= (\delta[n] - \alpha\delta[n-1]) * (\alpha^n u[n] - \alpha^n u[n-L]) \\
&= \delta[n] * \alpha^n u[n] - \alpha\delta[n-1] * \alpha^n u[n] - \delta[n] * \alpha^n u[n-L] + \alpha\delta[n-1] * \alpha^n u[n-L] \\
&= \alpha^n u[n] - \alpha\alpha^{n-1} u[n-1] - \alpha^n u[n-L] + \alpha\alpha^{n-1} u[n-1-L] \\
&= \alpha^n \big(\underbrace{u[n] - u[n-1]}_{=\delta[n]}\big) - \alpha^n \big(\underbrace{u[n-L] - u[n-1-L]}_{\delta[n-L]}\big) \\
&= \alpha^n \delta[n] - \alpha^n \delta[n-L] \\
&= \delta[n] - \alpha^L \delta[n-L]
\end{aligned}
$$

NOTE: $\delta[n-1] * x[n] = x[n-1]$

Non-zero only when $n=L$

d) The impulse response of the overall system is:

$$h[n] = \delta[n] - \alpha^L \delta[n-L]$$

so: $y[n] = x[n] - \alpha^L x[n-L]$

e) There is no finite value of L such that $y[n] = x[n]$ for $\alpha > 0$. However, if $\alpha < 1$ then $\alpha^L \to 0$ as $L \to +\infty$, so $y[n] = x[n]$ if $L = +\infty$, i.e. $h_2[n] = \alpha^n u[n]$.

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