PROBLEM:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

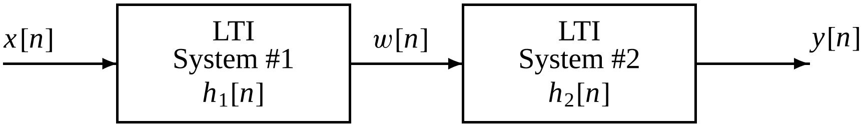

Figure 1: Cascade connection of two LTI systems.

(a) Suppose that LTI System #1 is described by the difference equation

$$w[n] = x[n] - \alpha x[n-1].$$

Determine the impulse response $h_1[n]$ of the first system.

(b) The LTI System #2 is described by the impulse response

$$h_2[n] = \alpha^n (u[n] - u[n-L]) = \sum_{k=0}^{L-1} \alpha^k \delta[n-k] = \begin{cases} \alpha^n & n = 0, 1, \ldots, L-1 \\ 0 & \text{otherwise.} \end{cases}$$

For the special case of $L = 6$, use convolution to show that the impulse response sequence of the overall cascade system is

$$h[n] = h_1[n] * h_2[n] = \delta[n] - \alpha^6 \delta[n-6].$$

(c) Generalize your result in part (b) for the general case of L any integer value.

(d) Obtain a single difference equation that relates $y[n]$ to $x[n]$ in Fig. 1.

(e) Assuming that $0 < \alpha < 1$, how would you choose L so that $y[n] = x[n]$ in Figure 1; i.e., how would you choose L so that the second system "undoes" the effect of the first system?

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.

SOLUTION

Part A

Plugging $x[n] = \delta[n]$ into the difference equation for system #1 yeilds

$$\boxed{h_1[n] = \delta[n] - \alpha\,\delta[n-1]}.$$

Part B

$$\begin{aligned}
h[n] &= h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k] h_2[n-k] \\
&= h_1[0]\,h_2[n-0] + h_1[1]\,h_2[n-1] \qquad (\text{since } h_1[k]{=}0 \text{ for } k{<}0 \text{ and } k{>}1) \\
&= h_2[n] - \alpha\,h_2[n-1] \\
&= \alpha^n\big(u[n] - u[n-6]\big) - \alpha\,\alpha^{(n-1)}\big(u[n-1] - u[n-7]\big) \\
&= \alpha^n\Big((u[n] - u[n-1]) - (u[n-6] - u[n-7])\Big) \\
&= \alpha^n(\delta[n] - \delta[n-6]) = \alpha^0\delta[n] - \alpha^6\delta[n-6] = \boxed{\delta[n] - \alpha^6\delta[n-6]}
\end{aligned}$$

Part C

$$\begin{aligned}
h[n] &= h_2[n] - \alpha\,h_2[n-1] \qquad (\text{from Part B}) \\
&= \alpha^n\big(u[n] - u[n-L]\big) - \alpha\,\alpha^{(n-1)}\big(u[n-1] - u[n-1-L]\big) \\
&= \alpha^n\Big((u[n] - u[n-1]) - (u[n-L] - u[n-1-L])\Big) \\
&= \alpha^n(\delta[n] - \delta[n-L]) = \alpha^0\delta[n] - \alpha^L\delta[n-L] = \boxed{\delta[n] - \alpha^L\delta[n-L]}
\end{aligned}$$

Part D

Replacing $\delta[n]$ with $x[n]$ and replacing $h[n]$ with $y[n]$ in the result of Part C yields

$$\boxed{y[n] = x[n] - \alpha^L x[n-L]}$$

Part E

If $0 < \alpha < 1$ then

$$\lim_{L\to\infty} \alpha^L = 0 \qquad \text{hence} \qquad \lim_{L\to\infty} h[n] = \delta[n] - 0 = \delta[n] \quad \text{and therefore} \quad \lim_{L\to\infty} y[n] = x[n]*\delta[n] = x[n].$$

Thus L should be chosen to be as large as possible (ideally $L = \infty$).

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