PROBLEM:

The system function of a linear time-invariant filter is given by the formula

$$H(z) = z^{-1}(1 - z^{-1})(1 - jz^{-1})(1 + jz^{-1})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. (*Hint: First multiply the factors to get a polynomial.*)

(b) What is the output if the input is $x[n] = \delta[n]$?

(c) Use multiplication of z-transform polynomials to find the output when the input is

$$x[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-3] + \delta[n-4].$$

(d) If the input to the system is of the form

$$x[n] = e^{j\hat{\omega}n} \qquad -\infty < n < \infty,$$

for what values of $\hat{\omega}$ will the output be zero for all n? We cannot use z-transforms directly to solve this problem, but we can find the frequency response from $H(z)$ and then solve the problem. Note that the factored form will tell you the answer and so will the pole zero plot for $H(z)$.

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SOLUTION

$$H(z) = z^{-1}(1-z^{-1})(1-jz^{-1})(1+jz^{-1})$$

(a) $$H(z) = (z^{-1}-z^{-2})(1+z^{-2}) =$$

$$= z^{-1} - z^{-2} + z^{-3} - z^{-4}$$

$$h(n) = \delta(n-1) - \delta(n-2) + \delta(n-3) - \delta(n-4)$$

$$y(n) = x(n-1) - x(n-2) + x(n-3) - x(n-4)$$

(b) $$h(n) = \delta(n-1) - \delta(n-2) + \delta(n-3) - \delta(n-4)$$

(c) $$x(n) = \delta(n) - 2\delta(n-1) + 2\delta(n-3) + \delta(n-4)$$

$$X(z) = 1 - 2z^{-1} + 2z^{-3} + z^{-4}$$

$$Y(z) = H(z)X(z) = [z^{-1} - z^{-2} + z^{-3} - z^{-4}][1 - 2z^{-1} + 2z^{-3} + z^{-4}]$$

$$\begin{array}{llllllll}
= z^{-1} & -2z^{-2} & & +2z^{-4} & +z^{-5} & & & \\
 & -z^{-2} & +2z^{-3} & & -2z^{-5} & -z^{-6} & & \\
 & & z^{-3} & -2z^{-4} & & +2z^{-6} & +z^{-7} & \\
 & & & -z^{-4} & +2z^{-5} & & -2z^{-7} & -z^{-8}
\end{array}$$

$$= z^{-1} - 3z^{-2} + 3z^{-3} - z^{-4} + z^{5} + z^{-6} - z^{-7} - z^{-8}$$

Then $$y(n) = \delta(n-1) - 3\delta(n-2) + 3\delta(n-3) - \delta(n-4) + \delta(n-5) + \delta(n-6)$$
$$-\delta(n-7) - \delta(n-8)$$

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(d) $x(n) = e^{j\hat{\omega} n} \qquad -\infty < n < \infty$

We need these $\hat{\omega}$ for which $\mathcal{H}(\hat{\omega}) = 0$

$$\mathcal{H}(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = e^{-j\hat{\omega}}\left[1 - e^{-j\hat{\omega}}\right]\left[1 - \underbrace{e^{j\pi/2}}_{=j} e^{-j\hat{\omega}}\right]\left[1 - \underbrace{e^{-j\pi/2}}_{=-j} e^{-j\hat{\omega}}\right]$$

$\mathcal{H}(\hat{\omega}) = 0$ for $1 - e^{-j\hat{\omega}} = 0 \Rightarrow \hat{\omega} = 0, \pm 2\pi, \pm 4\pi, \ldots$

Let's restrict $\hat{\omega}$ $-\pi < \hat{\omega} < \pi$ $\sim$ $\underline{\hat{\omega} = 0}$ is one solution.

$1 - e^{j\pi/2} e^{-j\hat{\omega}} = 0 \Rightarrow e^{j(\pi/2 - \hat{\omega})} = 1 = e^{j 2\pi m} \quad (m=0)$

$\pi/2 - \hat{\omega} = 2\pi m \Rightarrow \hat{\omega} = \frac{\pi}{2} - 2\pi m \quad m=0$ for $-\pi < \hat{\omega} < \pi$ $\sim$

$$\underline{\hat{\omega} = \pi/2}$$

$1 - e^{-j\pi/2} e^{-j\hat{\omega}} = 0 \Rightarrow e^{-j(\pi/2 + \hat{\omega})} = 1 = e^{j 2\pi m} \Rightarrow$

$\frac{\pi}{2} + \hat{\omega} = 2\pi m \Rightarrow$

$\hat{\omega} = 2\pi m - \frac{\pi}{2} \quad \sim \quad m=0$ for $-\pi < \hat{\omega} < \pi$

so $\underline{\hat{\omega} = -\pi/2}$

Summarizing, for $-\pi < \hat{\omega} < \pi$ the 3 possibilities are:

$$\hat{\omega} = 0, \frac{\pi}{2}, -\frac{\pi}{2}$$

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