PROBLEM:

We have developed several concepts that are useful in solving problems involving LTI systems. The main concepts are the *difference equation*, the *impulse response*, the *system function*, and the *frequency response* function. Most problem solving demands that you be able to go back and forth among these different mathematical representations of the LTI system because, as simple as it seems, the z-transform is *not* always the best tool for solving problems. Indeed, for a specific problem, one of these representations may be more convenient than the others, or we may need to use more than one of these representations in solving a given problem. The following is a simple problem that might be posed about an LTI system:

Given the input sequence $x[n]$ find the output sequence for all n when the system is an IIR filter:

$$y[n] = 0.8y[n-1] + x[n] + x[n-2].$$

The following is a partial list of possible approaches to solving this problem:

1. *Time-Domain:* Use the difference equation representation of the system to compute the output $y[n]$ for all required values of n. *For example, you could do this using* MATLAB.

2. *Z-Domain:* Multiply the z-transform of the input by the system function and determine $y[n]$ as the inverse z-transform of $Y(z)$.

3. *Frequency-Domain:* Break the input into a sum of complex exponential signals, use the frequency response function to determine the output due to each complex exponential signal separately, and finally, add the individual outputs together to get $y[n]$.

In each of these solution methods you would use one or more of the basic representations of the first-order IIR filter. Which method is easiest will have a lot to do with the nature of the input signal. For example, if you are given the difference equation and you want to use approach #2, you will have to determine the system function $H(z)$ from the difference equation coefficients.

Now in each of the following cases, the input will be given. In each case, determine which representation of the system and which of the above approaches will lead to the easiest solution of the problem, and detail the steps in using that approach to solve the problem. For example, if you choose approach #2 to solve the problem, your answer should be something like the following:

Step 1: Find $X(z)$, the z-transform of $x[n]$.

Step 2: Find $H(z)$, the system function of the first-order IIR filter.

Step 3: Multiply $X(z)H(z)$ to get $Y(z)$.

Step 4: Take the inverse z-transform of $Y(z)$ to get $y[n]$.

Now here are some possible inputs. In each case, state which of the approaches above (#1, #2, or #3) you would use. There may not be a clear cut answer. Give the approach that you *think* will yield the solution with least effort. Then carry out the method to get the output.

(a) $x[n] = u[n]$.

(b) $x[n] = 2\cos(0.5\pi n - \pi/2) + \cos(0.25\pi n - \pi)$ for $-\infty < n < \infty$.

(c) $x[n] = 10\delta[n-5]$.

(d) $x[n]$ is a sampled speech signal. It is represented by a vector of 10000 numbers. In this case, you do not have to find the actual output.

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458.

$$y(n) = 0.8y(n-1) + x(n) + x(n-2)$$

(a) $x(n) = u(n)$

Use z-transform

Step 1: $U(z) = \dfrac{1}{1-z^{-1}}$ $\quad |z|>1$

Step 2: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1+z^{-2}}{1-0.8z^{-1}}$

Step 3: $Y(z) = H(z)U(z) = \dfrac{1+z^{-2}}{1-0.8z^{-1}} \cdot \dfrac{1}{1-z^{-1}}$

Since numerator is of the same degree with denominator do the long division first:

$$\frac{1+z^{-2}}{(1-0.8z^{-1})(1-z^{-1})} = \frac{z^{-2}+1}{0.8z^{-2}-1.8z^{-1}+1} = \frac{2.25z^{-1}-0.25}{0.8z^{-2}-1.8z^{-1}+1} + 1.25$$

$$= 1.25 + \frac{2.25z^{-1}-0.25}{(1-0.8z^{-1})(1-z^{-1})}$$

Partial fraction expansion:

$$\frac{2.25z^{-1}-0.25}{(1-0.8z^{-1})(1-z^{-1})} = \frac{A}{1-0.8z^{-1}} + \frac{B}{1-z^{-1}}$$

$$A = \left.\frac{2.25z^{-1}-0.25}{1-z^{-1}}\right|_{z=0.8} = -10.25$$

$$B = \left.\frac{2.25z^{-1}-0.25}{1-0.8z^{-1}}\right|_{z=1} = 10$$

Then $Y(z) = 1.25 + \dfrac{-10.25}{1-0.8z^{-1}} + \dfrac{10}{1-z^{-1}} \Rightarrow$

$$y(n) = 1.25\delta(n) - 10.25(0.8)^n u(n) + 10u(n) =$$

$$= 1.25\delta(n) + [10 - 10.25(0.8)^n]\,u(n)$$

(b) $x(n) = 2\cos(0.5\pi n - \pi/2) + \cos(0.25\pi n - \pi) \qquad -\infty < n < +\infty$

Use frequency response

Find $\mathcal{H}(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-0.8z^{-1}} \sim \mathcal{H}(e^{j\hat{\omega}}) = \frac{1+e^{-j2\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

$$\mathcal{H}(e^{j(\hat{\omega}=\pi/2)}) = \frac{1+e^{-j2(\pi/2)}}{1-0.8e^{-j\pi/2}} = 0$$

$$\mathcal{H}(e^{j(\hat{\omega}=\pi/4)}) = \frac{1+e^{-j2(\pi/4)}}{1-0.8e^{-j\pi/4}} = \frac{1+e^{-j\pi/2}}{1-0.8e^{-j\pi/4}} = 1.983e^{-j0.5416\pi}$$

$$y(n) = 1.983\cos(0.25\pi n - \pi - 0.5416\pi) =$$

$$= 1.983\cos(0.25\pi n - 1.5416\pi) = 1.983\cos(0.25\pi n + 0.4584\pi)$$

(c) $x(n) = 10\,\delta(n-5)$

z-transform & impulse response

$$H(z) = \frac{1+z^{-2}}{1-0.8z^{-1}} = \frac{1}{1-0.8z^{-1}} + z^{-2}\frac{1}{1-0.8z^{-1}} \Rightarrow$$

$$h(n) = (0.8)^n u(n) + (0.8)^{n-2}u(n-2)$$

$$y(n) = h(n) * x(n) =$$

$$= h(n) * 10\,\delta(n-5) = 10\,h(n-5) =$$

$$= 10\left[(0.8)^{n-5}u(n-5) + (0.8)^{n-7}u(n-7)\right]$$

(d) Use MATLAB to calculate $y(n)$

either from $y(n) = 0.8y(n-1) + x(n) + x(n-2)$ for $y(0)=0$

or $y(n) = h(n) * x(n)$ since $h(n)$ is practically zero

for n large. For example $h(n=100) = 5.2199\cdot10^{-10}$

$h(n=200) = 1.0633\cdot10^{-19} \simeq 0$